Derived Schemes

Cailan Li

October 20th, 2020

1 Finite Presentation and Open Immersions

Definition 1.1. *A homomorphism of* SCR $R^\bullet \to S^\bullet$ *is locally of finite presentation if* $S^\bullet$ *is a compact object of* $\mathrm{SCR}_{R^\bullet}$*, given any filtered colimit* $\{B_i\}_{i\in J}$ *in* $\mathrm{SCR}_{R^\bullet}$ *we have*

$$\mathrm{Map}_{\mathrm{SCR}_{R^\bullet}}(S^\bullet, \varinjlim B_i) \cong \varinjlim \mathrm{Map}_{\mathrm{SCR}_{R^\bullet}}(S^\bullet, B_i)$$

Lemma 1.2. *Let* R_0 *be a discrete ring. Then the compact objects in* $R_0 - alg$ *are precisely the finitely presented* R_0-*algebras.*

Proof. Suppose $S = R_0[x_1, , \dots, x_n]/(f_1, \dots, f_m)$ is a finitely presented R_0-algebra. We first show that the natural map

$$\Phi : \varinjlim \mathrm{Hom}_{R_0}(S, B_i) \to \mathrm{Hom}_{R_0}(S, \varinjlim B_i)$$

is surjective. Aka given f on the RHS we need to show f factors through one of the factors B_i. Because our colimit is filtered, and we have finitely many generators, we can find a B_g for which $\{f(x_i)\}_{i=1}^n$ all live in. Because we have finitely many relations, we can find a further B_r for which $f_i(f(x_1), \dots, f(x_n)) = 0$ and thus we have an R_0 algebra map $S \to B_r$ as desired. Injectivity quickly follows from being finitely generated and therefore Φ is an isomorphism.

In the reverse direction, note that any R_0-algebra S is the colimit over it's finitely generated R_0-subalgebras S_i and this colimit is filtered. Thus we see that if S is compact,

$$\mathrm{Hom}_{R_0}(S, S) = \mathrm{Hom}_{R_0}(S, \varinjlim S_i) = \varinjlim \mathrm{Hom}_{R_0}(S, S_i)$$

Taking the identity map on the LHS, we get a splitting map, aka $\pi : S \to S_k$ s.t. $S \to S_k \xrightarrow{i} S$ is id_S for some k so that $S_k = S \oplus R$ as algebras. This will imply that S is finitely generated as writing the generators $x_i = x_i^s + x_i^r$ since $\pi : S_k \to S$ is an algebra homomorphism it follows that $x = \pi(x) = \pi(h(x_1, \dots, x_n)) = h(x_1^s, \dots, x_n^s)$. Thus we can write $S = R_0[x_1, \dots, x_n]/I$. To show I is finitely generated write $I = \varinjlim I_k$ as a colimit over finitely generated $R_0[x_1, \dots, x_n]$ submodules, aka subideals. Since colimits commute with colimits we have that

$$S = R_0[x_1, \dots, x_n]/I = \varinjlim R_0[x_1, \dots, x_n]/I_k$$

And now finitely presented follows by a similar argument as above.

Definition 1.3 (Open b/t affines)**.** *Let* $j : \mathrm{Spec}\, B^\bullet \to \mathrm{Spec}\, A^\bullet$ *be a morphism of affine derived schemes. We say that* j *is an open immersion if the corresponding morphism of* SCR *is locally of finite presentation, flat, and an epimorphism (*$B^\bullet \overset{L}{\otimes}_{A^\bullet} B^\bullet \to B^\bullet$ *is invertible)*

Definition 1.4 (Open target affine)**.** *Let $j : \mathscr{U} \to \operatorname{Spec} A^\bullet$ be a morphism of derived stacks. We say that j is an open immersion if it is a monomorphism and there is a (fpqc) covering family $\{\mathscr{U}_\alpha \to \mathscr{U}\}$ which induces an effective epimorphism $\sqcup_\alpha \mathscr{U}_\alpha \to \mathscr{U}$ (i.e. covers $\mathscr{U}$) such that each $\mathscr{U}_\alpha$ is affine and the composite $\mathscr{U}_\alpha \to \operatorname{Spec} A^\bullet$ is an open immersion of affine derived schemes.*

Remark. Let R be a discrete ring and let $\phi : R \to S$ be a ring homomorphism. Consider the cosimplicial set

$$(A_S^\bullet)_n = S \otimes_R \ldots \otimes_R S$$

where face and degeneracy maps were defined last week. Then the Amitsur complex will be

$$0 \to R \to (S \to (A_S^\bullet)_2 \to (A_S^\bullet)_3 \to \ldots)$$

If ϕ is faithfully flat, then $0 \to R \to A_S^\bullet$ will be a resolution.

Remark. Fiber products exist in derived stacks because $\operatorname{Sh}(X)$ is a Grothendieck $\infty-$topos (Lemma 6.2.2.7 in HTT) and therefore complete and cocomplete.

Definition 1.5. *A morphism of derived stacks $\mathcal{F} \to \mathcal{G}$ is an effective epimorphism if the canonical morphism*

$$\varinjlim_n \check{C}(\mathcal{F}/\mathcal{G})_n \to \mathcal{G}$$

is invertible where $\check{C}(\mathcal{F}/\mathcal{G})$ is the simplicial object with

$$\check{C}(\mathcal{F}/\mathcal{G})_n = \mathcal{F} \times_{\mathcal{G}} \ldots \times_{\mathcal{G}} \mathcal{F}$$

Lemma 1.6. *The notion of open with target affine $j : \mathscr{U} \to \operatorname{Spec} A^\bullet$ is preserved under base change by morphisms in* ADS*.*

Proof. Given $\operatorname{Spec} B^\bullet \to \operatorname{Spec} A^\bullet$, let $g^{-1}(\mathscr{U}) := \operatorname{Spec} B^\bullet \times_{\operatorname{Spec} A^\bullet} \mathscr{U}$, and now note we have the following cartesian diagram

$$\begin{array}{ccc}
g^{-1}(\mathscr{U}) \times_{\mathscr{U}} \bigsqcup \mathscr{U}_\alpha & \longrightarrow & \bigsqcup \mathscr{U}_\alpha \\
\downarrow & & \downarrow \\
g^{-1}(\mathscr{U}) & \longrightarrow & \mathscr{U} \\
\downarrow & & \downarrow \\
\operatorname{Spec} B^\bullet & \longrightarrow & \operatorname{Spec} A^\bullet
\end{array}$$

We now claim that $\left\{g^{-1}(\mathscr{U}) \times_{\mathscr{U}} \mathscr{U}_\alpha\right\}$ gives us our desired (fpqc) covering family so that $g^{-1}(\mathscr{U}) \to \operatorname{Spec} B^\bullet$ will be an open immersion. To see $\bigsqcup g^{-1}(\mathscr{U}) \times_{\mathscr{U}} \mathscr{U}_\alpha \to g^{-1}(\mathscr{U})$ is an effective epimorphism, first note that since geometric realization commutes with finite limits, we have that

$$\varinjlim_n g^{-1}(\mathscr{U}) \times_{\mathscr{U}} \check{C}\left(\bigsqcup \mathscr{U}_\alpha / \mathscr{U}\right)_n \to g^{-1}(\mathscr{U}) \tag{1}$$

is invertible. But note by transitivity of fiber products the Cartesian diagram above also shows

$$\bigsqcup g^{-1}(\mathscr{U}) \times_{\mathscr{U}} \mathscr{U}_\alpha \times_{g^{-1}(\mathscr{U})} \bigsqcup g^{-1}(\mathscr{U}) \times_{\mathscr{U}} \mathscr{U}_\alpha = \left(\bigsqcup g^{-1}(\mathscr{U}) \times_{\mathscr{U}} \mathscr{U}_\alpha\right) \times_{\mathscr{U}} \mathscr{U}_\alpha$$

so that in fact the Cech nerve of $\coprod g^{-1}(\mathscr{U}) \times_{\mathscr{U}} \mathscr{U}_\alpha \to g^{-1}(\mathscr{U})$ is exactly the LHS of Eq. (1) and thus is invertible. Now that each $g^{-1}(\mathscr{U}) \times_{\mathscr{U}} \mathscr{U}_\alpha$ is also affine again by transitivity of fiber products and therefore it suffices to show that an open immersion $\operatorname{Spec} C^\bullet \to \operatorname{Spec} A^\bullet$ of ADS is preserved under base change. Flatness for π_0 and epimorphism are clear, while finite presentation follows from tensor-hom adjunction. Thus, the only statement that is left to check is that given $\pi_*(A^\bullet) \otimes_{\pi_0(A^\bullet)} \pi_0(C^\bullet) \cong \pi_*(C^\bullet)$, we WTS

$$\pi_*(B^\bullet) \otimes_{\pi_0(B^\bullet)} \pi_0(B^\bullet \otimes_{A^\bullet} C^\bullet) \cong \pi_*(B^\bullet \otimes_{A^\bullet} C^\bullet)$$

But since $\pi_0(C^\bullet)$ is flat as a $\pi_0(A^\bullet)$ module and we see that

$$\otimes_{\pi_*(A^\bullet)} \pi_*(C^\bullet) = - \otimes_{\pi_*(A^\bullet)} \pi_*(A^\bullet) \otimes_{\pi_0(A^\bullet)} \pi_0(C^\bullet) = - \otimes_{\pi_0(A^\bullet)} \pi_0(C^\bullet) \tag{2}$$

and therefore $\pi_*(C^\bullet)$ is flat as a $\pi_*(A^\bullet)$ and thus the spectral sequence

$$E^2_{p,q} = \operatorname{Tor}_p^{\pi_*(A^\bullet)}(\pi_*(B^\bullet), \pi_*(C^\bullet))_q \implies \pi_{p+q}(B^\bullet \otimes_{A^\bullet} C^\bullet)$$

degenerates after E_2, and in fact is concentrated in the 0th column. It follows that $\pi_*(B^\bullet \otimes_{A^\bullet} C^\bullet) = \pi_*(B^\bullet) \otimes_{\pi_*(A^\bullet)} \pi_*(C^\bullet)$ and by degree considerations, $\pi_0(B^\bullet \otimes_{A^\bullet} C^\bullet) = \pi_0(B^\bullet) \otimes_{\pi_0(A^\bullet)} \pi_0(C^\bullet)$ and thus

$$\pi_*(B^\bullet) \otimes_{\pi_0(B^\bullet)} \pi_0(B^\bullet \otimes_{A^\bullet} C^\bullet) = \pi_*(B^\bullet) \otimes_{\pi_0(A^\bullet)} \pi_0(C^\bullet) \overset{\text{Eq. (2)}}{=\!=} \pi_*(B^\bullet \otimes_{A^\bullet} C^\bullet)$$

Definition 1.7 (Open in general)**.** *Let $j : \mathscr{U} \to \mathscr{X}$ be a morphism of derived stacks. We say j is an open immersion if for any affine derived scheme $\operatorname{Spec} R^\bullet$ and any morphism $\operatorname{Spec} R^\bullet \to \mathscr{X}$ we have that*

$$\begin{array}{ccc} \operatorname{Spec}(R^\bullet) \times_{\mathscr{X}} \mathscr{U} & \longrightarrow & \mathscr{U} \\ {\scriptstyle h}\downarrow & & \downarrow \\ \operatorname{Spec}(R^\bullet) & \longrightarrow & \mathscr{X} \end{array}$$

h is an open immersion in the above sense.

Remark. Definitions are consistent because of previous lemma.

2 Derived Schemes

Definition 2.1. *A Zariski cover of a derived stack $\mathscr{X}$ is a family $\{j_\alpha : \mathscr{U}_\alpha \to X\}$ where each j_α is an open immersion and the induced morphism*

$$\coprod_\alpha \mathscr{U}_\alpha \to \mathscr{X}$$

is an effective epimorphism.

Definition 2.2. *A derived stack $\mathscr{X}$ is called a derived scheme if it admits an affine Zariski Cover $\{j_\alpha : \mathscr{U}_\alpha \to X\}$ aka each $\mathscr{U}_\alpha$ is affine.*

Definition 2.3. *A classical scheme X is a derived scheme where we can find an affine Zariski cover $\operatorname{Spec} R_\alpha \to X$ where each R_α is discrete.*

Definition 2.4. *The inclusion* $c : \text{CRing} \hookrightarrow \text{SCR}$ *induces the inclusion* $\text{AS} \hookrightarrow \text{ADS}$*. So given a derived prestack* $\mathcal{X}$*,* $\mathcal{X}_{cl}$ *will be the presheaf on* CRing^{op}*.*

Recall we had an adjunction

$$\text{Hom}_{\text{ADS}}(\text{Spec}\, c(S), \text{Spec}\, R^\bullet) = \text{Hom}_{\text{Sch}}(\text{Spec}\, S, \text{Spec}\,(\pi_0(R^\bullet)))$$

But notice the LHS is just $(\text{Spec}\, R^\bullet)(c(S)) = \text{Spec}\,(R^\bullet)_{cl}(S)$ and so by Yoneda we actually have

$$\text{Spec}\,(R^\bullet)_{cl} \cong \text{Spec}\,(\pi_0(R^\bullet))$$

Lemma 2.5. *Given a derived stack* $\mathscr{X}$*, the presheaf* $\mathscr{X}_{cl}$ *will be a classical scheme and* $\mathscr{X} \mapsto \mathscr{X}_{cl}$ *is right adjoint to* i *: the inclusion of schemes into derived schemes.*

$$\text{Hom}_{\text{DSch}}(i(Y), \mathscr{X}) \cong \text{Hom}_{\text{Sch}}(Y, \mathscr{X}_{cl})$$

Remark. The counit of the adjunction above provides a natural map

$$i(\mathscr{X}_{cl}) \to \mathscr{X}$$

This map should be thought of as the analogue of the closed immersion of classical schemes

$$Y_{red} \hookrightarrow Y$$

In other words, $\mathscr{X}$ is an an infinitesimal thickening of $\mathscr{X}_{cl}$ except now the thickening lives in higher homotopy.

Warning. i does not preserve fiber products! In fact, this is a feature, not a bug of the theory of DAG, when you have two subvarieties V, W intersecting non-transversely, $[V \cap W]$ does not give you the correct intersection number. Using DAG, $[V \cap^h W]$ will be the correct answer instead.

Note that the other adjoint $\mathscr{X} \to \mathscr{X}_{cl}$ does preserve fiber products however.

We can therefore generate a lot of examples of truly derived schemes by taking the derived fiber product of classical schemes.

Example 1. Consider the SCR $\Bbbk \otimes^L_{\Bbbk[x]} \Bbbk$ viewed as a derived scheme. Because $_{cl}$ preserves fiber products, we see that the underlying classical scheme is $\text{Spec}\, \Bbbk \otimes_{\Bbbk[x]} \Bbbk$ which topologically is a point. A very useful explicit description of this SCR can be gotten by noting that $\Bbbk$ has a Koszul resolution $\Lambda(\Bbbk[1]) \otimes \Bbbk[x]$

$$(0 \to \Bbbk[x] \xrightarrow{1 \mapsto x} \Bbbk[x]) \xrightarrow{x \mapsto 0} \Bbbk \to 0$$

as exterior algebras (well, technically the Sullivan algebras) are exactly the cofibrants object in $\text{cdga}^{\geq 0}$ (under projective model structure) which is Quillen equivalent to SCR in characteristic 0 via monoidal Dold-Kan. Therefore we see that

$$\Bbbk \otimes^L_{\Bbbk[x]} \Bbbk = (\Lambda(\Bbbk[1]) \otimes \Bbbk[x]) \otimes_{\Bbbk[x]} \Bbbk = \Lambda(\Bbbk[1]) \otimes \Bbbk$$

2.1 Alternative Definitions

Definition 2.6. *Given a topological space X, let* $\mathrm{SCR}(X)$ *be the subcategory of stacks/sheaves in* $[\mathrm{Op}(X), \mathrm{SCR}]$. *Then the* $\infty-$ *category* dRgSp *of derived ringed spaces are pairs* $(X, \mathcal{O}_X^\bullet)$ *where X is a topological space and* $\mathcal{O}_X^\bullet \in \mathrm{SCR}(X)$. *Then* dRgSp^{loc} *will then be the nonfull* $\infty-$*subcategory where* $(X, \pi_0(X^\bullet))$ *is a locally ringed space.*

Definition 2.7. *The* $\infty-$*category of derived schemes is the full subcategory of* dRgSp^{loc} *consisting of objects* $(X, \mathcal{O}_X^\bullet)$ *s.t.*

(1) The truncation $(X, \pi_0(X^\bullet))$ *is a scheme.*

(2) $\forall\, i$, *the sheaf of* $\pi_0(\mathcal{O}_X^\bullet)-$*modules* $\pi_i(X^\bullet)$ *is quasi-coherent.*

Example 2. With this notion of derived scheme, we see that after Dold-Kan, the corresponding $\mathcal{O}_X^\bullet$ for $\Bbbk \otimes^L_{\Bbbk[x]} \Bbbk$ will be the dga $(\Lambda(\Bbbk[1]) \otimes \Bbbk, 0)$. Under Dold-Kan homotopy groups correspond to homology and thus

$$\pi_0(\mathcal{O}_X^\bullet) = H^0(\Lambda(\Bbbk[1]) \otimes \Bbbk) = \Bbbk, \quad \pi_1(\mathcal{O}_X^\bullet) = H^1(\Lambda(\Bbbk[1]) \otimes \Bbbk) = \Bbbk, \quad \pi_i(\mathcal{O}_X^\bullet) = 0 \ \forall i > 1$$

Thus we see that the classical scheme $(\mathrm{Spec}\, \Bbbk, \Bbbk)$ does not detect the higher homotopy $\pi_1(\mathcal{O}_X^\bullet) = \Bbbk$.

2.2 Structure Sheaf of $A^\bullet$

For the purposes of having a reference, for k characteristic 0, it turns out we can further replace $\mathrm{cdga}_k^{\geq 0}$ with the category of connective E_∞ $k-$algebras ($k-$algebras with a commutative and associative multiplication up to coherent homotopy), up to quasi-isomorphism. This model (spectral algebraic geometry) has the following theorem

Proposition (Prop 1.1.4.3 in SAG). *Let $A^\bullet$ be an E_∞ ring. Then $\exists$ a sheaf of E_∞ rings $\mathscr{O}_{A^\bullet}$ on the topological space* $\mathrm{Spec}\, \pi_0(A^\bullet)$ *and a morphism* $\phi : A^\bullet \to \mathscr{O}_{A^\bullet}(\mathrm{Spec}\, \pi_0(A^\bullet))$ *which satisfies the following conditions*

(a) $\forall x \in \pi_0(A^\bullet) = R$ *consider the distinguished open* U_x. *Then the composition map* $A^\bullet \xrightarrow{\phi} \mathscr{O}_{A^\bullet}(\mathrm{Spec}\, R) \to \mathscr{O}_{A^\bullet}(U_x)$ *induces an equivalence of* E_∞ *rings* $A^\bullet[x^{-1}] \to \mathscr{O}_{A^\bullet}(U_x)$.

(b) Taking π_0 *of* ϕ, *the canonical map*

$$R = \pi_0(A^\bullet) \to \pi_0(\mathscr{O}_{A^\bullet}(\mathrm{Spec}\, R)) \to (\pi_0 \mathscr{O}_{A^\bullet})(\mathrm{Spec}\, R)$$

is an isomorphism and thus $\pi_0(\mathscr{O}_{A^\bullet}) \cong \mathcal{O}_{\mathrm{Spec}\, R}$.

Theorem (Thrm 7.5.0.6 in HA). *Given an* E_∞ *ring* $A^\bullet$ *and an étale morphism of commutative rings* $\pi_0(A^\bullet) \to B$, *there exists an* E_∞ *ring* $B^\bullet$*(unique up to equivalence) and an étale map of* E_∞ *rings* $f : A^\bullet \to B^\bullet$ *and an isomorphism of* $\pi_0(A^\bullet)-$*algebras* $\pi_0(B^\bullet) \cong B$.

Remark. Applying Thrm above to $\pi_0(A^\bullet) \to \pi_0(A^\bullet)[x^{-1}]$(which is étale as geometrically corresponds to the open immersion $U_x \hookrightarrow \mathrm{Spec}\, R$) in Proposition above, this produces our E_∞ ring $A^\bullet[x^{-1}]$.

Therefore, any SCR $A^\bullet$ naturally gives rise to a derived scheme $(\mathrm{Spec}\, \pi_0(A^\bullet), \mathscr{O}_{A^\bullet})$ in the above sense.

Example 3. Concretely, given a cdga A and $f \in H^0(A)$, we have that

$$\mathscr{O}_A(U_f) = A[\widetilde{f}^{-1}]$$

where $\widetilde{f} \in A_0$ is any lift of f (Assume you know how to localize a cdga). Different choices of lifts give different algebras, but they are all quasi-isomorphic.